.Problema para ajudar na escola: Fração trigonométrica

Problema

(A partir da 1ª série do E. M.- Nível de dificuldade: Difícil**)**

Seja θ um ângulo agudo tal que

$$A = \frac{2\,\text{sen}^3\,\theta \cdot \cos\,\theta}{\text{sen}\,\theta + \cos\,\theta - 1} + \frac{2\cos^3\theta \cdot \text{sen}\,\theta}{\text{sen}\,\theta + \cos\,\theta + 1}$$

e

$$B = \cos\,\theta\,(1 + \cos\,\theta) + \text{sen}\,\theta\,(1 + \text{sen}\,\theta) - 1\,.$$

Verifique que $\boxed{\dfrac{A}{B} = 1 + \text{sen}\,\theta - \cos\,\theta}$.

Adaptado da **V ONEM**, 2008.

Lembretes

Relação fundamental da trigonometria:

$\text{sen}^2\,\alpha + \cos^2\,\alpha = 1$, para qualquer medida angular α.

Fatoração da diferença de dois quadrados/ Produto da soma pela diferença:

$\boxed{m^2 - n^2 = (m+n)\cdot(m-n)}$, para quaisquer $m, n \in \mathbb{R}$.

Solução

Antes de mais nada, observe que θ é um ângulo agudo, ou seja, $0 < \theta < 90^\circ$. Assim, $\text{sen}\,\theta \neq 0;\ \cos\,\theta \neq 0;\ \text{sen}\,\theta \neq \pm 1$ e $\cos\,\theta \neq \pm 1$.

Para simplificar as contas, vamos trabalhar separadamente com o numerador e o denominador da razão $\dfrac{A}{B}$.

► Numerador:

$$A = \frac{2\,\text{sen}^3\,\theta \cdot \cos\,\theta}{\text{sen}\,\theta + \cos\,\theta - 1} + \frac{2\cos^3\theta \cdot \text{sen}\,\theta}{\text{sen}\,\theta + \cos\,\theta + 1}$$

$$A = 2\,\text{sen}\,\theta \cdot \cos\,\theta \left(\frac{\text{sen}^2\,\theta}{\text{sen}\,\theta + \cos\,\theta - 1} + \frac{\cos^2\theta}{\text{sen}\,\theta + \cos\,\theta + 1}\right)$$

$$A = 2\,\text{sen}\,\theta \cdot \cos\,\theta \left(\frac{\text{sen}^2\,\theta \cdot (\text{sen}\,\theta + \cos\,\theta + 1) + \cos^2\theta \cdot (\text{sen}\,\theta + \cos\,\theta - 1)}{(\text{sen}\,\theta + \cos\,\theta - 1)\cdot(\text{sen}\,\theta + \cos\,\theta + 1)}\right)\ .\quad (i)$$

Antes de prosseguir, observe que:

$$\begin{aligned}
\boxed{(\text{sen}\,\theta + \cos\,\theta - 1)(\text{sen}\,\theta + \cos\,\theta + 1)} &= ((\text{sen}\,\theta + \cos\,\theta) - 1)((\text{sen}\,\theta + \cos\,\theta) + 1)\\
&= (\text{sen}\,\theta + \cos\,\theta)^2 - 1^2\\
&= \sin^2\,\theta + 2(\sin\,\theta)(\cos\,\theta) + \cos^2\,\theta - 1\\
&= (\text{sen}^2\,\theta + \cos^2\,\theta) + 2\,\text{sen}\,\theta \cdot \cos\,\theta - 1\\
&= 1 + 2\,\text{sen}\,\theta \cdot \cos\,\theta - 1\\
&= \boxed{2\,\text{sen}\,\theta \cdot \cos\,\theta}\ .\quad (ii)
\end{aligned}$$

Substituindo (ii) em (i), segue que:

$$A = 2\,\text{sen}\,\theta \cdot \cos\,\theta \left(\frac{\text{sen}^2\,\theta \cdot (\text{sen}\,\theta + \cos\,\theta + 1) + \cos^2\theta \cdot (\text{sen}\,\theta + \cos\,\theta - 1)}{2\,\text{sen}\,\theta \cdot \cos\,\theta}\right)$$

$$A = \cancel{2\,\text{sen}\,\theta \cdot \cos\,\theta} \left(\frac{\text{sen}^2\,\theta \cdot (\text{sen}\,\theta + \cos\,\theta + 1) + \cos^2\theta \cdot (\text{sen}\,\theta + \cos\,\theta - 1)}{\cancel{2\,\text{sen}\,\theta \cdot \cos\,\theta}}\right)$$

$$A = \text{sen}^2\,\theta \cdot (\text{sen}\,\theta + \cos\,\theta + 1) + \cos^2\theta \cdot (\text{sen}\,\theta + \cos\,\theta - 1)$$

$$A = \text{sen}^2\,\theta \cdot (\text{sen}\,\theta + \cos\,\theta) + \text{sen}^2\,\theta + \cos^2\theta(\text{sen}\,\theta + \cos\,\theta) - \cos^2\theta$$

$$A = (\text{sen}\,\theta + \cos\,\theta) \cdot \left(\text{sen}^2\,\theta + \cos^2\theta\right) + \text{sen}^2\,\theta - \cos^2\theta$$

$$A = (\text{sen}\,\theta + \cos\,\theta) \cdot 1 + \text{sen}^2\,\theta - \cos^2\theta$$

$$A = \text{sen}\,\theta + \cos\,\theta + \left(\text{sen}^2\,\theta - \cos^2\theta\right)$$

$$A = (\text{sen}\,\theta + \cos\,\theta) + (\text{sen}\,\theta + \cos\theta)\cdot(\text{sen}\,\theta - \cos\theta)$$

$$\boxed{A = (\text{sen}\,\theta + \cos\,\theta)\cdot(1 + \text{sen}\,\theta - \cos\theta)}\ .\quad (iii)$$

► Denominador:

$$B = \cos\,\theta\,(1 + \cos\,\theta) + \text{sen}\,\theta\,(1 + \text{sen}\,\theta) - 1$$

$$B = \cos\,\theta + \cos^2\,\theta + \text{sen}\,\theta + \text{sen}^2\,\theta - 1$$

$$B = \cos\,\theta + \text{sen}\,\theta + \left(\cos^2\,\theta + \text{sen}^2\,\theta\right) - 1$$

$$B = \cos\,\theta + \text{sen}\,\theta + 1 - 1$$

$$\boxed{B = \cos\,\theta + \text{sen}\,\theta}\ .\quad (iv)$$

Finalmente, por (iii) e por (iv) concluímos que,

$$\frac{A}{B} = \frac{(\text{sen}\,\theta + \cos\,\theta)\cdot(1 + \text{sen}\,\theta - \cos\theta)}{\cos\,\theta + \text{sen}\,\theta}$$

$$\frac{A}{B} = \frac{\cancel{(\text{sen}\,\theta + \cos\,\theta)}\cdot(1 + \text{sen}\,\theta - \cos\theta)}{\cancel{\cos\,\theta + \text{sen}\,\theta}}$$

$$\frac{A}{B} = 1 + \text{sen}\,\theta - \cos\theta\,.$$

Assim, de fato, $\boxed{\dfrac{A}{B} = 1 + \text{sen}\,\theta - \cos\theta}$.

Solução elaborada pelos **Moderadores do Blog**.